LIQUIDITY PREFERENCE UNDER UNCERTAINTY: A MODEL OF
DYNAMIC INVESTMENT IN ILLIQUID ASSETS

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Abstract

The paper presents a mathematical model of optimal investment in illiquid assets. Specifically, the model addresses the problem of an investor with limited capital resources, who makes sequential decisions on long-term investments under uncertainty as to future opportunities. The model demonstrates that such an investor will optimally demand a higher rate of return on investments in long-term, unmarketable assets and that the size of the premium demanded is an increasing function of the duration of the illiquid investment.

Liquidity Preference under Uncertainty: A Model of Dynamic Investment
in Illiquid Assets

I. Introduction

The article presents a mathemetical model of liquidity
preference in an uncertain environment.   The model addresses the problem
of an investor, with access to a limited pool of capital, who makes
sequential decisions on long-lasting investments, under uncertainty as
to future opportunities. Our results demonstrate that a rational
investor will, under these circumstances, demand a higher rate of
return (liquidity premium) on investments in long-lasting unmarketable assets
than on marketable ones. The liquidity premium demanded is an increasing
function of the duration of the illiquid investment.

Lumpiness and limited reversibility are important characteristics of
some types of investment decisions (particularly investments in real capital).
Yet, with few exceptions, financial models of optimal investment
(portfolio allocation) and the equilibrium structure of returns, rest on
the assumption that asset markets are fluid and frictionless. That is,
it is generally assumed that no transaction costs or trading indivisibilities
exist which would hamper individuals in their initial allocation of
wealth or in the free revision of their portfolios, and it is extremely
difficult to incorporate such effects (and retain analytic tractability)
into these models. For this reason, our analysis begins at a different point:
from the outset we define some investment opportunities to be illiquid,
and from there proceed to consider optimal investment criteria for such
assets (in a dynamic context, where accepting one illiquid investment may
cause the investor to forfeit a better one in the future).

Liquidity, in this analysis, is considered to be a time variable:

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the illiquidity of an asset is measured by the (probable) amount of
time the asset locks up capital, and prevents the investor from taking
advantage of new opportunities that might appear.[1] Thus, in the
literature of asset choice, our model follows the lines of Arrow [1968],
Henry [1974], and Hirschleifer [1971]. In this paper we consider the
effect of this type of illiquidity on a binary (all-or-nothing)
investment process. In a subsequent paper, we shall consider optimal
decision rules for a _portfolio_ of liquid and illiquid assets.

Our model differs from Hirschleifer's in its formulation (as a
sequential decision process rather than a time-state preference model)
and from the fact that liquidity premia arise, not on account of consumer
preference for level consumption, but because of the possibility of loss
from foregone investment opportunities. We differ from Arrow and
Henry in considering, instead of permanently irreversible decisions,
choices between investments lasting varying amounts of time. We
believe a particularly useful contribution of our model is that the
formulation clearly demonstrates the dependence of the optimal decision
rule on the relative durations of investment alternatives and on the
return distribution characterizing the opportunity set: we are not aware
of such a result's existence elsewhere in the literature.

Section 2 presents basic assumptions and notation. In Section 3,
we formulate a model of sequential investment under uncertainty, deriving
optimal policies for both finite and infinite horizons. In Section 4, we
examine the dependence of the optimal decision rule and the investor's
welfare on certain parameters characterizing the dynamic process; in

particular, we consider the effect of an exogenous shift in short-term
rates of return on investment in long-term assets. Section 5 presents
our conclusions and indicates potential lines of future research.

II. Assumptions and Notation

Investment Opportunities. In order to gain insight into the nature of
the decision process, we seek to restrict the portfolio and opportunity
state variables of our model to as few as possible, while still
conserving the basic structure of the illiquid investment problem.
Therefore, let us consider a world having two types of indivisible
assets: an "illiquid" long-term asset and a "liquid" short-term asset
the latter of which may be costlessly traded at any time. The two assets
always have a cost per unit equal to the investor's total available funds,
thus only one asset at a time may be held. The investor has no options
other than these two assets in the allocation of his wealth; if he elects
not to invest in the long term asset he must hold the short-term asset.
His purpose in holding these assets is to generate a stream of returns for
consumption.

In what follows, we shall consistently distinguish between "investments"
and "opportunities." Opportunities in our model are potential capital
investments, which the investor perceives, but has not yet acted upon. An
opportunity becomes an investment if and only if the investor acts to accept
it (commits resources to it).

Opportunities to make long-term investments arrive according to a

Poisson process with characteristic frequency λ; the average inter-arrival time between opportunities is $\tau = 1/\lambda$. A long-term investment, once purchased, matures according to a Poisson process with frequency $\mu(\mu < \lambda)$; the average duration of a long term investment is therefore $T = 1/\mu \; (T > \tau)$.

Each opportunity (potential long-term investment) has a characteristic rate of return x (expressed in dollars per unit time), representing the height of a level stream which the investor would receive until maturity. Rates of return on successive opportunities are independent, identically distributed (i.i.d.); they are drawn from a known probability density function $f(x)$. The rate of return on a given opportunity is known to the investor at the time the opportunity arrives and not before. The rate of return on the liquid (short-term) asset is arbitrarily set to zero, thus x measures the incremental rate of return on an illiquid investment, over and above what could be obtained by investment in the liquid asset.

The rate of return x represents the rate at which income is generated over the life of the investment. All income is consumed by the investor as it is received and may not be used to increase capital available for new investments. Capital, representing the capacity to undertake new investments, is returned to the investor only when an investment expires (matures) and not before. Thus capital is strictly conserved.

Discussion. The assumption that long-term opportunities mature probabilistically (according to a Poisson process with expected duration T) may at first appear superfluous, but is, in fact, an important feature of the formulation. Essentially, the assumption provides a way for time to pass within the model in a memoriless fashion (thereby avoiding the need

to keep track of the age of any asset). We should point out, however, that the results of the model are not restricted to this formulation. In particular, analysis of a parallel model, in which time passes in fixed-length intervals (interval τ between arrivals of opportunity, interval T between acceptance and maturation of an investment) yields identical asymptotic results for the steady-state. Unfortunately, the fixed-interval formulation is difficult to generalize to portfolio problems (involving 2 or more assets).[2]

The assumptions that returns on the long-term asset are (1) certain (after arrival) (2) realized as a continuous stream over the lifetime of the asset and (3) necessarily consumed when received were made in order to facilitate the direct comparison of returns (in terms of the relative heights of two cash flow streams) on long- and short-term investments. None of these assumptions are inherent to the formal model, which can accommodate stochastic and irregularly timed returns on long-term invest-ments as long as (1) the investor is able to judge the net present value of an opportunity when it arrives and (2) for any opportunity the interval between acceptance and the time the investor is free to make a new invest-ment has a Poisson distribution with characteristic parameter μ.

Investor's Objective Function. We will assume that the investor seeks to maximize his total undiscounted additive dollar return over a finite horizon, and his average rate of return over an infinite horizon.[3] Mathematically, the formulation permits analysis with or without discounting.[4] We chose to consider undiscounted returns because, by assumption, x represents a pure consumption stream for which no alternate investment opportunities exist. Alternative use of funds, the usual justification of discounting,

has thus been eliminated from the model; discounting in this context
would only be justified by pure time preference for consumption on the
part of the investor (a much weaker rationale). Lacking a strong justifi-
cation for discounting returns, we elected, for ease of exposition, to
concentrate on the no-discounting formulation: however, the reader should
note that the analysis may be extended straightforwardly to the case of
discounted returns (with a discount factor less than one).[5]

    <u>Decision Process and Decision Rule</u>. We have formulated the illiquid
opportunities investment problem in continuous time, as a semi-Markov
decision process with rewards. In problems of this type it is convenient if
the process converges over time so that optimization takes place over finite
quantities. Our procedure, therefore, is to formulate the functional equation
of the optimization problem assuming that the investor discounts future rewards
by a factor $e^{-\alpha t}$ $(\alpha > 0)$. We then take the limit of the functional
equation in continuous time, letting $\alpha \to 0$, thereby ensuring conformity
with the ergodic theorems and thus convergence of the undiscounted
optimization problem.[6] It is important to emphasize that, because we have
assumed that the investor is indifferent to the timing of returns, the
"discount rate" α is simply a mathematical construct, which vanishes
in the limit and has no significance for the economic analysis.[7]

    In analyzing the investor's decision problem, we consider maximization
of the objective function by means of a decision rule of the form: accept
opportunity x if and only if x is greater than or equal to ξ , where ξ ,
the minimum or "hurdle" rate of return is constrained to be constant, and

is determined by the parameters λ and μ, the probability distribution function $f(x)$ and the time t remaining before the horizon. For a finite horizon, it has been shown that a "constant" decision rule is not strictly optimal; the investor's optimal hurdle rate is a function of time, and optimization involves solving for the function $\xi(t)$ which maximizes the objective function between now and the horizon (over the finite interval $[t,0]$). However, it also has been shown that as the horizon becomes long the best constant hurdle rate converges to the truly optimal hurdle rate for the steady state.[8]

Notation. The notation introduced above is summarized as follows:

(1) λ --arrival time of opportunities (Poisson)

  $\tau = 1/\lambda$ --average interarrival time between opportunities

(2) μ --departure rate of investments (Poisson)

  $T = 1/\mu$ --average maturation time for one investment $(T > \tau)$

(3) $\tilde{x}$ --uncertain rate of return on an illiquid opportunity/investment

  x --a realization of the random variable $\tilde{x}$:

$$f(x)dx = Pr[x \leq \tilde{x} \leq x + dx]$$

  ξ --a hurdle rate in the decision process

  ξ^* --optimal constant hurdle rate such that opportunity x is accepted if and only if $x \geq \xi^*$

(4) α --continuous time "discount rate" applicable to future returns to ensure convergence $(\alpha \rightarrow 0$ in the limit)

(5) t --time remaining to the horizon $(t \rightarrow \infty$ in the limit)

In addition to the above, we will use the following:

(6) $F(\xi) \equiv \int_0^{\xi} f(x)dx$

  $G(\xi) \equiv 1 - F(\xi) = \int_{\xi}^{\infty} f(x)dx$

$$r(\xi) \equiv \int_{\xi}^{\infty} xf(x)\,dx$$

$$E(\tilde{x} \mid \tilde{x} \geq \xi) = \frac{r(\xi)}{G(\xi)}$$

$$\gamma(\xi) \equiv \mu + \lambda G(\xi)$$

We shall not hesitate to drop the functional argument of $F(\xi)$, $G(\xi)$, $r(\xi)$, and $\gamma(\xi)$, writing F, G, r, and γ wherever convenient.

III. Model Formulation and Analysis

Value of an Opportunity. Consider an opportunity drawn from the distribution $f(x)$. After it is drawn, the investor is able to assess the present value of the cash flow stream: for the assumed discount rate α and departure rate μ, the present value of a continuous, constant stream x to a risk-neutral investor is:

$$\int_{0}^{\infty} \left[\int_{0}^{t} xe^{-\alpha s}\,ds \right] \mu e^{-\mu t}\,dt = \frac{x}{\mu+\alpha} \tag{1}$$

If the investor accepts only those opportunities such that $x \geq \xi$,

then, looking forward at the future, the investor can calculate the conditionally expected present value of an acceptable opportunity (conditional on the decision rule: accept if $x \geq \xi$); this quantity is:

$$\frac{\displaystyle\int_{\xi}^{\infty} \frac{x}{\mu+\alpha} f(x)\,dx}{\displaystyle\int_{\xi}^{\infty} f(x)\,dx} = \frac{r(\xi)}{(\mu + \alpha)G(\xi)} \tag{2}$$

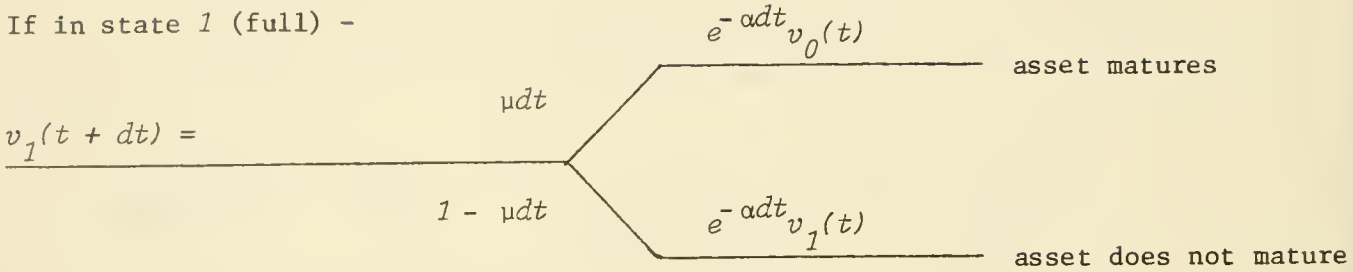Decision Process: State Space and Decision Tree Representation. Under the assumption of all-or-nothing investment, the investor's portfolio may at any time be in one of two states: state 0 ("empty") or state 1 ("full"). Let $v_0(t)$ represent the present value of starting in state 0 with time t left to go before the horizon, if a constant policy (based on a hurdle rate ξ) is followed. Similarly, let $v_1(t)$ represent the value of starting in state 1 with t left to go, following a constant policy. We are interested in the sequential decision process for long or infinite horizons, therefore, it may be assumed that the values of terminating in state 0 or state 1 are identically zero:

$$v_0(0) = v_1(0) = 0.$$

For the infinitesimal interval dt preceding time t, the investor faces a decision problem which can be represented in tree form as follows:

If in state 0 (empty) –

If in state *1* (full) -

$$v_1(t + dt) = $$

μdt $e^{-\alpha dt} v_0(t)$ asset matures

$1 - \mu dt$ $e^{-\alpha dt} v_1(t)$ asset does not mature
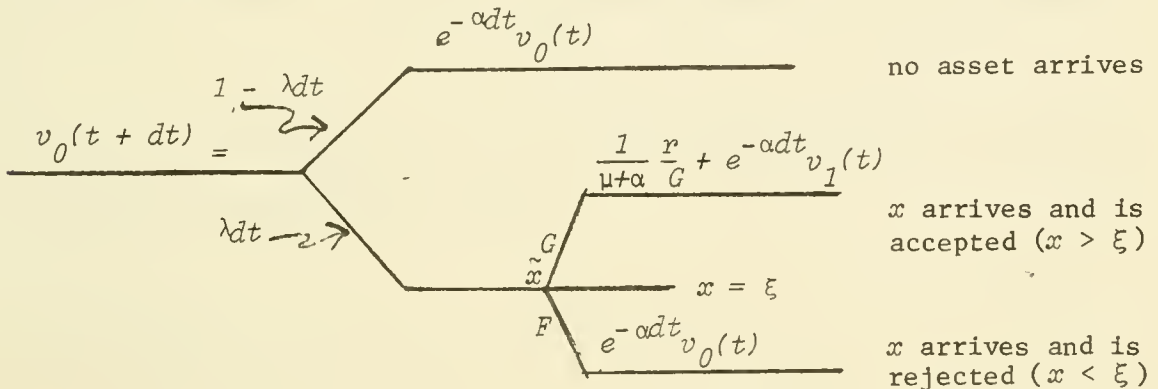
<u>Discussion.</u> The above decision trees have straightforward interpretations: nevertheless, some comments may serve to clarify subsequent analysis. First, the reader should note that time runs backward, i.e., the investor counts time by units left to go. Loosely speaking therefore, an investor poised at time *3dt* anticipates events and decisions which will take him to time *2dt*, subsequently to time *1dt* and finally to time *0* (his horizon, the point at which the <u>decision</u> process terminates).

The reader should also note that a reward is earned from the process in interval *dt* if and only if (1) the portfolio is <u>empty</u> (state *0*) at $t + dt$; (2) a long-term investment opportunity <u>arrives</u> during the interval; and (3) the rate of return clears the hurdle rate ξ so that the investor <u>accepts</u> (invests in) the opportunity. According to the formulation, if an opportunity is accepted, the investor immediately (1) earns the present value of the investment $\frac{1}{\mu + \alpha} \cdot \frac{r}{G}$ and (2) switches from portfolio state *0* (empty) to *1* (full) until the investment matures. Thus the formulation accounts for all

rewards resulting from the acceptance of an opportunity at the time the opportunity is accepted, rather than during the time the investor holds an earning asset (state $\bar{1}$, when the portfolio is full) and actual returns are realized.

It is evident that, in the decisional context, state 1 serves only to keep track of the passage of time: as long as the portfolio is full the investor can take no action with respect to new opportunities which may arrive. On the other hand, if the portfolio is empty (state 0), the arrival of an opportunity makes necessary a decision to accept or reject it. In so deciding the investor should consider the investment's returns over its expected life: mathematically his calculation of future benefit is taken into account be concentrating the expected present value of the opportunity at the instant of decision.

It must be emphasized that in making investment decisions, the investor only looks forward to potential future opportunities, never backward at what actually happened to his investments. Because asset lives are uncertain, the actual returns realized by the investor may (in fact are certain to) deviate from expectations. However, since actual returns are not reinvested, such deviations only serve to hasten or delay the arrival of future decision points. Regardless of such deviations in the timing, at points in time when investment decisions arise (triggered by the arrival of a new opportunity), the entire past history of the portfolio is irrelevant to the decision to be made. Information about actual performance of the portfolio may thus be suppressed in the formulation (thereby greatly simplifying the state-space representation of the dynamic process).

Functional Equations. The functional equations for the decision process summarized in tree form above are:

$$v_0(t + dt) = [1 - \lambda dt + \lambda F dt]e^{-\alpha dt}v_0(t)$$

$$+ [\lambda G dt] \ e^{-\alpha dt}v_1(t) + [\lambda G dt] \ \frac{1}{\mu + \alpha} \ \frac{r}{G} \tag{3a}$$

$$v_1(t + dt) = [\mu dt] \ e^{-\alpha dt}v_0(t) + [1 - \mu dt] \ e^{-\alpha dt}v_1(t) \tag{3b}$$

If we ignore terms of higher order than dt , then substitution of the expanded form of $e^{-\alpha dt}$ into (3a) and (3b), and multiplication yields

$$v_0(t + dt) = [1 - \lambda G dt - \alpha dt]v_0(t)$$

$$+ [\lambda G dt]v_1(t) + [\lambda G dt] \ \frac{1}{\mu + \alpha} \ \frac{r}{G} \tag{4a}$$

$$v_1(t + dt) = [\mu dt]v_0(t) + [1 - \mu dt - \alpha \ dt] \ v_1(t) \tag{4b}$$

Subtracting $v_0(t)$ from both sides of Eq. (4a) and $v_1(t)$ from both sides of Eq. (4b), dividing through by dt, taking the limit as $dt \to 0$, and then the limit as $\alpha \to 0$ obtains:

$$\frac{dv_0(t)}{dt} = [- \lambda G]v_0(t) + [\lambda G]v_1(t) + \frac{\lambda}{\mu} r \tag{5a}$$

$$\frac{dv_1(t)}{dt} \;=\; [\mu]v_0(t) \,+\, [-\mu]v_1(t) \qquad\qquad (5b)$$

(N.B.: Analysis of discounted and undiscounted cases is identical until we passed to the limit $\alpha \to 0$, but diverges from this point on.)

Equations (5a) and (5b) describe a system of two linear, constant-coefficient differential equations. We may write Eqs. (5a) and (5b) in matrix notation as follows:

$$\frac{d}{dt}\,V(t) \;=\; \frac{\lambda}{\mu}\,R \,+\, AV(t) \qquad\qquad (6)$$

where

$$V(t) \;\equiv\; \begin{Bmatrix} v_0(t) \\ v_1(t) \end{Bmatrix} \qquad R \;\equiv\; \begin{Bmatrix} r \\ 0 \end{Bmatrix}$$

and A is a 2 x 2 differential matrix of transition probabilities:

$$A \;\equiv\; \begin{bmatrix} -\lambda G & \lambda G \\ \mu & -\mu \end{bmatrix}$$

Solution by Exponential Transform. The system described by (6) above

may be solved via an exponential transform.[8] Let $E(s)$ denote the
exponential transform of the vector $V(t)$:

$$E(s) \equiv \int_0^\infty V(t)\ e^{-st}dt$$

Then the exponential transform of Eq. (6) may be taken and solved to obtain
the transformed solution (we have already assumed that the reward for
terminating in either state is identically zero):

$$E(s) = \frac{1}{s}\ (sI - A)^{-1}\ \frac{\lambda}{\mu}\ R\ . \tag{7}$$

Inverting $(sI - A)$, multiplying the resultant by the scalar $\frac{1}{s}$, and
expanding by partial fractions yields:

$$\frac{1}{s}\ (sI - A)^{-1} = \frac{1}{s^2 \gamma} \begin{bmatrix} \mu & \lambda G \\ \mu & \lambda G \end{bmatrix} + \frac{1}{\gamma^2}\ (\frac{1}{s} - \frac{1}{s + \gamma}) \begin{bmatrix} \lambda G & -\lambda G \\ -\mu & \mu \end{bmatrix} \tag{8}$$

(recall $\gamma \equiv \mu + \lambda G$). Substitution of Eq. (8) into Eq. (7) and application
of the inverse transform yields expressions for total expected future
values of states 0 and 1 as functions of the parameters μ and λ ,
the probability density function $f(x)$ (implicit in G, r, and γ)
and the time remaining until the horizon:

$$v_0(t) = \frac{\lambda}{\mu} r \left[\frac{\mu}{\gamma} t + \frac{\lambda G}{\gamma^2} (1 - e^{-\gamma t}) \right] \tag{9a}$$

$$v_1(t) = \frac{\lambda}{\mu} r \left[\frac{\mu}{\gamma} t + \frac{\mu}{\gamma^2} (1 - e^{-\gamma t}) \right] \tag{9b}$$

Eq. (9a) and (9b) may be analyzed to find the investor's best constant decision rules for finite horizons; however, since such rules are known to be suboptimal, we proceed directly to consideration of the infinite-horizon or steady-state optimal policy.

Steady State Decision Rule. As the investor's horizon approaches infinity, the Markov process on returns approaches a steady state. In the steady state, the investor's expected "gain" (return per unit time) from starting in either state 0 (empty) or state 1 (full) must be the same. Dividing Eqs. (9a) and (9b) through by t and taking limits as $t \to \infty$, it is seen that the $(1 - e^{-\gamma t})$ term in each expression becomes insignificant, thus:

$$\lim_{t \to \infty} \frac{v_0(t)}{t} = \lim_{t \to \infty} \frac{v_1(t)}{t} = g = \frac{\lambda}{\gamma} r \tag{10}$$

The investor seeks to maximize his expected gain g with respect to the hurdle rate ξ. Taking the derivative of the gain with respect to ξ yields:

$$\frac{dg}{d\xi} = \frac{f(\xi)}{\gamma^2} \left[\lambda r - \xi \gamma \right] . \tag{11}$$

The derivative $dg/d\xi$ is strictly positive if $\xi = 0$, and it is easy to

prove (see Appendix A) that (1) there exists one and only one ξ^* such that $\lambda r - \gamma \xi^* = 0$; and (2) for all $\xi > \xi^*$, $\lambda r - \gamma \xi < 0$. Thus the investor's gain has a <u>unique maximum</u> when the hurdle rate ξ is

$$\xi^* = \frac{\lambda r^*}{\gamma^*} \tag{12}$$

The righthand side of Eq. (12) is simply (see Eq. (10)) the rate of return (per unit time) the investor may expect on average upon application of the optimal decision rule "accept opportunity x if and only if $x \geq \xi^*$." Not surprisingly, in a steady state environment, the investor's expected gain attains a maximum at the point where <u>marginal</u> (hurdle) and <u>average</u> rates of return are equal.

The optimal hurdle rate ξ^* is strictly greater than zero for all $\lambda > 0$, as long as the probability of encountering opportunities having rates of return greater than zero is positive. ξ^* is the minimum rate of return the investor should optimally demand to commit his wealth to a long-term (illiquid) investment. Since the rate of return on alternative short-term investments has been scaled to zero, ξ^* should be interpreted as a <u>liquidity</u> premium; that is, as an <u>excess rate of return</u> which must be earned on long-term unmarketable assets in order to compensate the investor for the opportunity loss of foregoing possibly more attractive investments during the time his capital is locked up.

<u>Finite Horizon Decision Rule.</u> We now consider the dependence of optimal hurdle rates on the time remaining to the horizon. Referring back to the original decision tree formulation, define $v_0^*(t)$ as the expected value of starti

in state 0 with time t left to go if a truly optimal (non-constant) policy is followed. Similarly, define $v_1{}^*(t)$ as the value of starting in state 1 with t left to go, following an optimal policy. Substituting $v_0{}^*(t)$ and $v_1{}^*(t)$ for $v_0(t)$ and $v_1(t)$ in the decision tree and functional equations (3a) and (3b), it is evident that all derivations and observations with respect to $v_0(t)$ and $v_1(t)$ are obtained identically for $v_0{}^*(t)$ and $v_1{}^*(t)$. In particular we obtain the equation system:

$$\frac{dv_0{}^*(t)}{dt} = \underset{\xi(t)}{Max} \left\{ \left[-\lambda G\right] v_0{}^*(t) + \left[\lambda G\right] v_1{}^*(t) + \frac{\lambda}{\mu} r \right\} \qquad (13a)$$

$$\frac{dv_1{}^*(t)}{dt} = [\mu]\, v_0{}^*(t) + [-\mu] v_1{}^*(t) \qquad (13b)$$

describing the optimization at time t of the Markov process in continuous time. Note, however, that optimization now takes place with respect to the <u>function</u> $\xi(t)$ governing hurdle rates over the time remaining to the horizon.

In Appendix B, it is shown that the optimal instantaneous hurdle rate applicable at time t (the hurdle rate function) satisfies:

$$\xi^*(t) = \frac{\alpha\,(t)\lambda\,r^*}{\gamma^*} = \alpha\,(t)\,\xi^* \qquad (14)$$

where r^*, γ^*, and ξ^* are optimal values of r, γ, and ξ <u>in the steady state</u> and $\alpha\,(t)$ is a monotonically increasing concave function of t such that $0 \le \alpha\,(t) \le 1$. From (14) it is evident that with respect to time, $\xi^*(t)$ behaves like $\alpha(t)$: that is, optimal hurdle rates are small (near 0) for very short horizons, but increase at a decreasing rate as the investor's

horizon lengthens. Naturally, as the horizon becomes infinite, optimal
hurdle rates approach asymptotically the constant (steady state) optimal
hurdle rate ξ^* derived in the previous section.

II. Comparative Statics

Dependence of ξ^* and g on τ and T . In this section we consider how the opti
hurdle rate ξ^* and investor's welfare (measured by his average gain g) are
affected by changes in the process parameters τ (interarrival time between
opportunities) and T (average duration of long-term investments). Our
analysis is limited to the infinite horizon (steady state) decision process,
but the results extend to finite horizon processes as well.

In Equation (12) above, the optimal (steady state) hurdle rate ξ^*
shown to be an implicit function of τ and T, as well as the return dis-
tribution $f(x)$. Equation (12) may be re-stated as:

$$\frac{r^*}{\xi^*} - G^* = \frac{\tau}{T} ; \qquad (12')$$

It is straightforward to show that $\frac{r}{\xi} - G$ is a monotonically decreasing
function of ξ; the RHS of (12') is constant. Thus, as Figure 1 indicates,
the optimal ξ^* is uniquely determined by the intersection of $\frac{r}{\xi} - G$ and $\frac{\tau}{T}$.

Figure 1
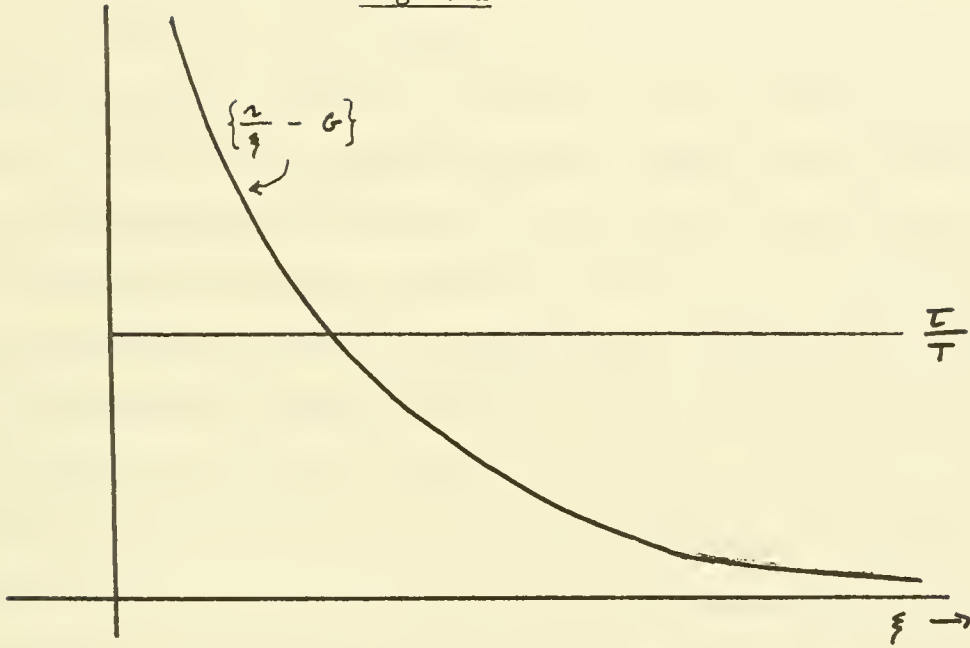
Looking at Figure 1, it is evident that raising τ or lowering T raises the intersection point, thus <u>lowers</u> the optimal hurdle rate ξ^*; conversely, lowering τ or raising T raises the optimal ξ^*. In Equations (10) and (12) above it was shown that, at the optimum, the investor's hurdle rate ξ^* equals the average (expected) gain g; thus raising T or lowering τ makes the investor <u>better off</u>, whereas lowering T or raising τ makes him <u>worse off</u>.

<u>Hurdle Rates as a Function of Asset Duration</u>. We have just shown that, all other things equal, optimal hurdle rates are an increasing function of T, the expected duration of a long-term investment. Optimization within the il-liquid opportunities process thus determines a functional relationship between <u>necessary</u> rates of return and the (expected) duration of investments. The relationship indicates that rational investors demand higher returns on un-marketable assets of longer duration.

Although the functional relation between rate of return and duration is broadly consistent with a "liquidity preference" hypothesis, the function $\xi^*(T)$ which specifies the relation between required rate of return and maturity, is not equivalent to the term structure of interest rates which would be observed in the capital markets. In the first place, $\xi^*(T)$ is not a directly observable function. The hurdle rate ξ^* (for T given) is only a <u>lower bound</u> defining a <u>minimally acceptable</u> rate of return on unmarketable investments of average duration T; rates of return on actual investments made under this decision rule could have any value above ξ (within the domain of $f(x)$). A second and more subtle difference between

$\xi^*(T)$ and a market term structure equation, is that <u>it is not an indifference relation</u>: the equality of ξ^* and g implies that the investor is better off for higher ξ^* and T. Equilibrium in the term structure can occur only if a representative investor would be indifferent between alternatives of different maturities. It therefore makes sense to ask: What changes in the rate of return and/or interarrival distributions would cause an investor, <u>on average</u>, to be indifferent to sampling opportunities from among different maturity classes? Such a line of inquiry lies outside the scope of this research, but appears promising for the future development of a theory of equilibrium term structure in real capital markets with little or no secondary trading.

<u>Impact of Short-Term Rates on Long-Term Investment.</u> We conclude by considering the effect of an exogenous rise in <u>short-term (market) rates of return</u> on "real" investment (i.e. irreversible physical capital and/or long-lasting projects).

Let ξ_0^* be the hurdle rate (liquidity premium) applicable to an illiquid investment opportunity of expected duration T for a known market rate Z. Let ξ_δ^* be the hurdle rate (liquidity premium) applicable to the same illiquid opportunity given market rate $Z + \delta$. From Equation (12) above, it is known that:

$$\xi_0{}^* = \frac{\lambda r}{\gamma} = \frac{Tr}{TG + \tau} \; .$$

(15)

(Optimization is understood, thus stars (*) have been suppressed except for the hurdle rate $\xi_\theta{}^*$.)

We can solve for the hurdle rate $\xi_\delta{}^*$ by rescaling returns x by δ :

$$x' = x - \delta \; ,$$

(16)

and applying Equation (12) to the transformed problem: (primes denote quantities rescaled according to (16)):
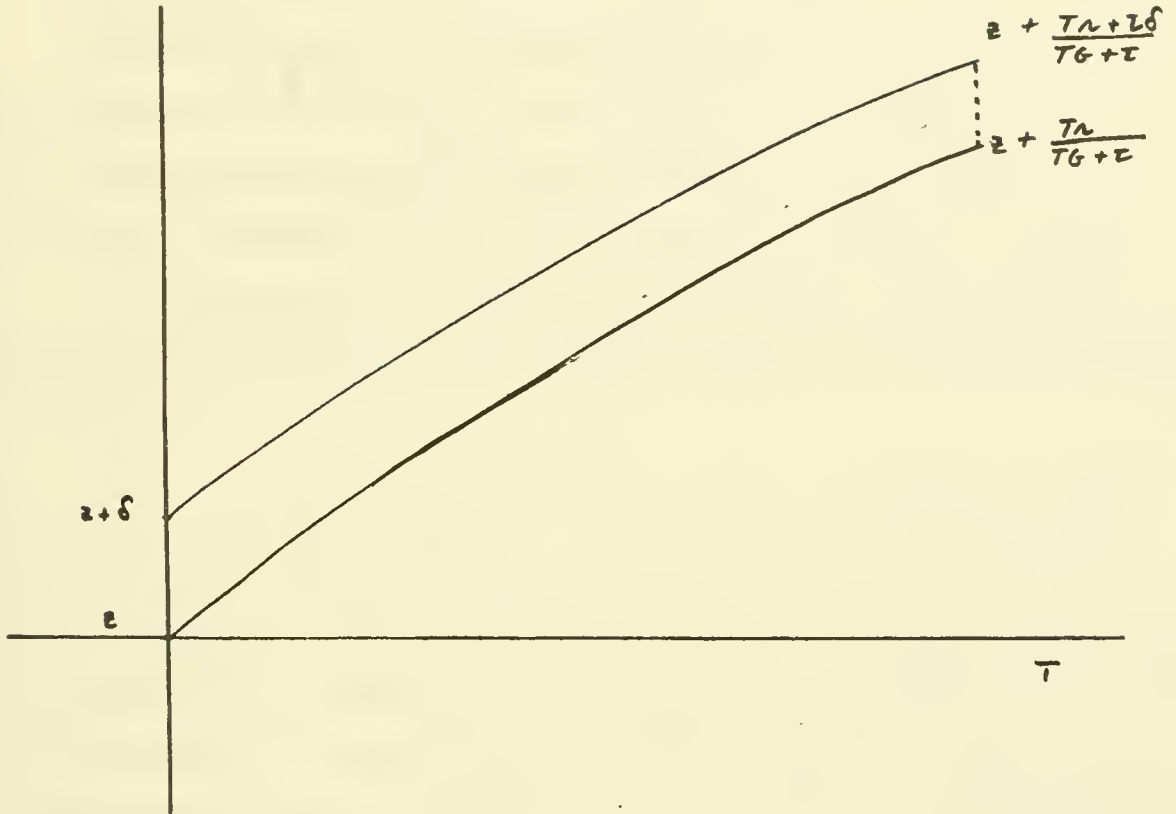
$$\xi_\delta{}^{*\prime} = \frac{Tr'}{TG' + \tau} \; .$$

(17)

Reverse transformation of (17) obtains $\xi_\delta{}^*$ in units comparable to (15):

$$\xi_\delta{}^* = \frac{Tr + \tau\delta}{TG + \tau} > \xi_0{}^* \; .$$

(18)

Comparing (18) to (15), it is evident that an exogenous rise in "short" or "market" rates of return raises the <u>absolute hurdle rate</u> on new long-term investments. Figure 2 illustrates the change in hurdle rates for assets with maturities ranging from 0 to T .

hurdle
rate

Figure 2

Given an unchanging opportunity set, a rise in the short rate from Z to $Z + \delta$ will cause a corresponding (but less than proportional) rise of the long (hurdle) rate from $Z + \frac{Tr}{TG + \tau}$ to $Z + \frac{Tr + \tau\delta}{TG + \tau}$. An increased hurdle rate implies that the investor optimally accepts fewer long-term opportunities. The model thus supports the Keynesian hypothesis: ceteris paribus higher rates of return on short-term financial instruments dis-courage entrepreneurial investment in new, long-term productive opportunities.

IV. Conclusion.

The illiquid opportunities model, described and analyzed above, addresses the problem of an investor, with limited capital (or limited access to capital markets), who makes sequential decisions on long-lasting investments under uncertainty as to future opportunities. The model was mathematically formulated as a continuous-time Markov decision process with rewards. Specific restrictive assumptions necessary to the formulation included (1) a linear, additive-separable investor's utility function over time; (2) investment opportunities of a standard frequency and duration drawn independently from a known stationary distribution $f(x)$; and (3) indivisible investment of all or none of the investor's wealth in any opportunity. Under these simplified assumptions we explored (1) conditions under which it is rational for an investor to prefer liquid investments and to demand extra return (a "liquidity premium") on long-lasting illiquid investments, and (2) functional characteristics of the liquidity premia demanded. We were able to show, in the context of the model, that an investor facing sequential opportunities to make long-lasting investments will optimally demand of such investments positive liquidity premia which are an increasing function of their (probable) duration.

The model appears to open numerous lines of future research. In the
first place, the formulation, although attractive in that its analytic
results are tractable and intuitive, is unnecessarily restrictive. In
particular, the assumption that the representative investor is limited
to all-or-nothing investment policies may be relaxed: generalization of
the "illiquid opportunities" model to the portfolio context will be treated
in a subsequent article.

Further interesting questions arise if the representative investor
is considered as a social aggregate. Society as a whole has at any time
limited resources available to invest in new opportunities; of these oppor-
tunities, certain investmenst in productive assets or infrastructure changes
generally return their value over a relatively long period of time. Thus,
both centrally planned and free market economies face the problem which our
model poses: in allocating resources between short- and long-term real in-
vestments (tangible assets and productive technology), the benefits of a
current opportunity must always be balanced against potential future losses
from foregone or delayed opportunities. In this context, work along the
lines described in Section 4 above may provide a characterization of term
structure equilibrium based on dynamic aspects of the real capital investment
process.

In conclusion, the general problem of liquidity relates
to both transaction _time_ and transaction _cost_, and realistically involves
interdependence between the two. The "illiquid opportunities" model
presented here considers only temporal aspects of liquidity
preference. Despite its inherent limitations, it is

hoped that the model's results offer insight into the concept of liquidity as well as into the general principles by which rational investors may decide to commit resources to irreversible investments.

Appendix A

Existence of a Unique ξ^* (Optimal Steady State Hurdle Rate).

Theorem. (1) For given λ, μ (both positive) and probability density function $f(x)$ [defined on $(0,\infty)$], there exists one and only one ξ^* (with associated values of r^*, G^*, γ^*) such that

$$\lambda r^* - \gamma^*\xi^* = 0 \ .$$

(2) For all $\xi > \xi^*$, $\lambda r - \gamma\xi < 0$.

Proof. At $\xi = 0$

$$\lambda r - \gamma\xi = \lambda \int_0^\infty x \ f(x) \ dx - 0 > 0 \ . \qquad \text{(A-1)}$$

As $\xi \to \infty$, $\lambda r \to 0$, $\gamma \to \mu$, thus

$$\lim_{\xi \to \infty} \lambda r - \gamma\xi = -\infty < 0 \ . \qquad \text{(A-2)}$$

However, $\lambda r - \gamma\xi$ is strictly decreasing, since the derivative of $\lambda r - \gamma\xi$ with respect to ξ is always negative:

$$\frac{\partial}{\partial \xi} (\lambda r - \gamma\xi) = -\lambda f(\xi)\xi + \lambda f(\xi)\xi - \gamma = -\gamma < 0 \ . \qquad \text{(A-3)}$$

Therefore, $\lambda r - \gamma\xi = 0$ for **exactly** one ξ^* on $(0,\infty)$ and $\lambda r - \gamma\xi < 0$ for all $\xi > \xi^*$.

Appendix B

Analysis of Finite Horizon Hurdle Rates

In this appendix we consider the behavior of optimal (non-constant) hurdle rates, when the time remaining to the horizon is <u>finite</u>. The two lemmas which follow are necessary for our analysis of the optimal hurdle rate function $\xi^*(t)$.

<u>Lemma 1.</u> The function $v_0^*(t) - v_1^*(t)$ is a positive, monotonically increasing function of t on the interval $(0, \infty)$.

<u>Proof.</u> Let $\xi_t^* \equiv \arg \max v_0^*(t + dt)$. We may then take the difference of the recursive equations (4a) and (4b):

$$v_0^*(t + dt) - v_1^*(t + dt) = [1 - \lambda G(\xi_t^*)dt - \mu dt - \alpha dt][v_0^*(t) - v_1^*(t)] + \frac{\lambda r(\xi_t^*)dt}{\mu + \alpha} . \tag{B-1}$$

Note that optimization is implicit for $G(\bullet)$ and $r(\bullet)$.

Consider the process with one instant left to go. Recall that, by assumption $v_0^*(0) = v_1^*(0) = 0$, and $r(\bullet)$ and $G(\bullet)$ are positive for all ξ, thus:

$$v_0^*(dt) - v_1^*(dt) = \frac{\lambda r(\xi_0^*)dt}{\mu + \alpha} > v_0^*(0) - v_1^*(0) = 0. \tag{B-2}$$

Now consider the process with two instants left to go. Substituting

from (B-2) into (B-1) and ignoring terms of order $[dt]^2$ or higher

we obtain:

$$v_0{}^*(2dt) - v^*(2dt) \overset{\backsim}{=} \frac{\lambda}{\mu + \alpha} \; [r(\xi_1{}^*)dt + r(\xi_0{}^*)dt] \qquad \text{(B-3)}$$

$$> v_0{}^*(dt) - v_1{}^*(dt) \; .$$

By successive substitution into the recursion equation (always ignoring terms

of higher order than dt) we obtain for the process with $m+1$ instants left

to go:

$$v_0{}^*[(m + 1)dt] - v_1{}^*[(m + 1)dt] = \frac{\lambda}{\mu + \alpha} \; [r(\xi_m^*)dt + \qquad \text{(B-4)}$$

$$r(\xi_{m-1}^*)dt + \; . \; . \; . \; + r(\xi_0{}^*)dt]$$

From (B-4) it is obvious that $v_0^* - v_1{}^*$ is (1) positive and (2) strictly

increasing for all m (therefore t) on the interval $(0, \infty)$.

Lemma 2. Let r^* and G^* respectively denote functions r and G evaluated for

the optimal steady state hurdle rate ξ^* (see Eq. 12). Then:

$$\underset{t \to \infty}{lim} \; [v_0{}^*(t) - v_1{}^*(t)] = \frac{\lambda r^*}{\mu \, G^*} \quad .$$

Proof. In the steady state, <u>marginal</u> returns to the "v^* process" must equal

the (per period) gain from following the optimal constant policy:

$$\lim_{t \to \infty} \frac{dv_i{}^*(t)}{dt} = \max_{\xi} \lim_{t \to \infty} \frac{v_i(t)}{t} = g^* = \frac{\lambda r^*}{\gamma^*} \; ; \quad i = 0, 1 \tag{B-5}$$

Note: g^* and γ^* denote g and γ evaluated at ξ^*.

But $dv_i{}^*/dt$ must also satisfy differential equations (13a) and (13b), in the limit as $t \to \infty$ for the constant optimal policy $\xi^* = \xi(t)$:

$$\lim_{t \to \infty} \frac{dv_0{}^*(t)}{dt} = \lim_{t \to \infty} \left\{ -\lambda G^* [v_0{}^*(t) - v_1{}^*(t)] + \frac{\lambda}{\mu} r^* \right\} \tag{B-6a}$$

$$\lim_{t \to \infty} \frac{dv_1{}^*(t)}{dt} = \lim_{t \to \infty} \left\{ \mu [v_0{}^*(t) - v_1{}^*(t)] \right\} \tag{B-6b}$$

Substituting from (B-5) into the LHS of (B-6a) or (B-6b) and rearranging allows us to deduce the limiting value of $v_0{}^*(t) - v_1{}^*(t)$:

$$\lim_{t \to \infty} [v_0{}^*(t) - v_1{}^*(t)] = \frac{\lambda r^*}{\mu G^*} \tag{B-7}$$

[Recall that G^* and r^* appear as "constants" in (B-6a) and (B-6b), thus are not affected by the limit passage in t]. Q.E.D.

We are now equipped to characterize optimal finite-horizon hurdle rates (the function $\xi^*(t)$) by the following:

Theorem. In the illiquid opportunities process, with time t left to go, the optimal hurdle rate $\xi^*(t)$, applicable to new investment opportunities, satisfies:

$$\xi^*(t) = \alpha(t)\xi^*$$

where ξ^* is the optimal steady state hurdle rate, and $\alpha(t)$ is a monotonically increasing, concave function of t such that $0 \le \alpha(t) < 1$, $\forall\, t\varepsilon(0,\infty)$.

Proof. From Lemmas 1 and 2, we have that $v_0^*(t) - v_1^*(t)$ is a monotonically increasing function of t which approaches a finite upper bound as $t \to \infty$. Therefore, we may scale values of the function, for finite values of t:

$$v_0^*(t) - v_1^*(t) = \alpha(t)\frac{\lambda r^*}{\mu\gamma^*}\ , \tag{B-8}$$

where $\alpha(t)$ is (for now) a monotonically increasing function of t whose range is $(0,1)$.

Substituting for $v_0^* - v_1^*$ in Eq. (13a), and carrying out the maximization, we have:

$$\xi^*(t) = \frac{\alpha(t)\lambda r^*}{\gamma^*} = \alpha(t)\xi^*\ . \tag{B-9}$$

That $\alpha(t)$ and $\xi^*(t)$ are both concave functions of t may be seen from Eqs. (B-4), (B-8), and (B-9). Eq. (B-9) implies for Eq. (B-4) that $\xi^*_m > \xi^*_{m-1}$, thus $r(\xi^*_m) < r(\xi^*_{m-1})$. This result indicates that increments to $v_0^* - v_1^*$ are decreasing as m(therefore t) increases. Thus $v_0^* - v_1^*$ is concave in t, therefore [from (B-8) and (B-9)] $\alpha(t)$ and $\xi(t)$ are also concave in t.   Q.E.D.

Footnotes

(1)   Some researchers have preferred to define liquidity in terms of tranaction costs and/or liquidation penalties. In the context of single-period Capital Asset Pricing Model, Chen, Kim and Kon [1975] derived investor demand functions and equilibrium asset valuation formulae for the case where investors experience random, end-of-period cash demands, and incur proportional liquidation costs on the sale of risky assets. Using a stochastic control formulation, Magill and Constantinides [1976] obtained dynamic portfolio selection rules for an investor who incurs proportional tranactions costs on purchases and sales of securities. A somewhat different approach is that of Goldman [1974, 1978]; he considers the case in which an isolated investor allocates his portfolio among bonds whose interest rates and value on liquidation before maturity are inversely related, and shows that given uncertainty about future interim consumption needs, the consumer optimally holds a diversified portfolio. Tobin [1958] has proposed that liquidity be defined as a functional relationship between transactions cost and the time necessary to convert an asset into cash.

(2)   In addition to simplifying the model's formulation, the assumption of uncertain duration, we believe, reflects uncertainty which actually exists about the anticipated useful life of many tangible and intangible long-term assets. For example, a corporation or entrepreneur, introducing a new product, is never perfectly certain of the product's life expectancy; only the probable duration of its lifespan is known when the investment is made.

(3) The theoretical limitations of an objective function which is additive
in time are quite severe; Meyer's [1976] research into the mathematical
structure of time preference seems to indicate that (individual or corporate)
preferences with respect to time cannot be adequately represented as the sum
of discounted returns or even as the sum of additively-separable single-period
utilities (the most common intertemporal utility function found in the economic
literature). Despite prior recognition of the deficiencies of the objective
function, we include it in the formulation; redefinition of x in terms of
utility rates of return allows the model to encompass additive separable
utility functions.

(4) In a private communication Richard Grinold has provided us with an
analysis of the case where the investor discounts future returns.

(5) Cf. A. Veinott [1969].

(6) Cf. D. Blackwell [1962]; Sheldon Ross [1970] pp. 162-163.

(7) However, α takes on economic significance if we assume the investor's
rate of time preference (α) to be the same as the market opportunity
rate (currently not specified). In this case the rate earned on short-term
investments would be α; $\alpha + x$ would be earned on long-term investments.
Formulation of the illiquid investment problem under these assumptions
would be identical to that shown except for passage to the limit in α.

(8) Cf. Blackwell [1962]; Jeremy Shapiro [1968].

(9) Cf. Howard [1972], Vol. II, pp. 797 ff.

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1031-78